Proofs of matrix properties on p. 16:

i) We claim that $A^*Ax = 0 \Leftrightarrow Ax = 0$. Proof of $\Leftarrow$ is trivial. In the opposite direction we have $A^*Ax = 0 \Rightarrow x^*A^*Ax = 0 \Rightarrow (Ax)^* Ax = 0 \Rightarrow Ax = 0$. This automatically implies $r(A^*A) = r(A)$ because if a set of columns in A is linearly dependent then the same set of columns in A^*A must be linearly dependent and vice versa.

ii) $Bx = 0 \Rightarrow ABx = 0$ which means $r(B) \geq r(AB)$ (if a set of columns in B is dependent then the same set of columns is dependent in AB). On the other hand each column in AB is a linear combination of columns in A and therefore the span of columns in AB cannot be greater than the span of columns in A. Since $\mathrm{Span}(A) \supseteq \mathrm{Span}(AB)$, by the dimension theorem $r(A) = \dim \mathrm{Span}(A) \geq \dim \mathrm{Span}(AB) = r(AB)$.

iii) Now by virtue of i) and ii) we have $r(A^*A) = r(A) \leq r(A^*)$. Reversing the role of A and A^* we also have $r(AA^*) = r(A^*) \leq r(A)$. The two inequalities can only hold if $r(A) = r(A^*)$.

iv) By the dimesionality theorem $r(A) \leq m$ and $r(A^*) \leq n$. By virtue of iii) we then obtain $r(A) = r(A^*) \leq \min(m, n)$.