Proof of Theorem 2.11

1. The "if" part.

   (a) Define the subspace of all pay-offs with non-positive price,
   $$M_- := \{Ax : x \in \mathbb{R}^n, S^* x \leq 0\},$$
   and let X_+ be the positive orthant
   $$X_+ := \{y \in \mathbb{R}^m : y_i \geq 0 \text{ for } i = 1, \ldots, m\}.$$
   If there is no arbitrage (type I or type II) then
   $$M_0 \cap X_+ = \{\mathbf{0}\}.$$
   There are two aspects of X_+ that make it unsuitable for direct application. Firstly, it intersects M_0, albeit at only one point. Secondly it is unbounded. Consider the intersection of X_+ with the hyperplane $H(\mathbf{1}, 1) := \{y \in \mathbb{R}^m : \sum_{i=1}^m y_i = 1\}$. We have that $X_+ \cap H(\mathbf{1}, 1)$ is a closed and, unlike X_+, bounded convex set. Furthermore,
   $$M_0 \cap (X_+ \cap H(\mathbf{1}, 1)) = \text{empty set}.$$

   (b) We can now apply a separation theorem for two disjoint convex sets, one of which (M_0) is closed and the other one ($X_+ \cap H_1$) is compact (in finite dimension compact = closed and bounded), see Beavis and Dobbs (1990) . This means there is a hyperplane $H(\psi, c) := \{y \in \mathbb{R}^m : \sum_{i=1}^m \psi_i y_i = c\}$ such that
   $$\begin{aligned} \psi^* y &\leq c \text{ for all } y \in M_-, \\ \psi^* y &> c \text{ for all } X_+ \cap H(\mathbf{1}, 1). \end{aligned} \tag{1}$$

   (c) Because M_- contains $M_0 := \{Ax : x \in \mathbb{R}^n, S^* x = 0\}$ and M_0 is a linear subspace (i.e. $y \in M_0 \Rightarrow -y \in M_0$) it must be the case that $c = 0$. We can interpret $H(\tilde{\psi}, c)$ as the zero price subspace of a *completed market* with a state price vector proportional to $\tilde{\psi}$. The inequality
   $$\tilde{\psi}^* y > 0 \text{ for all } X_+ \cap H(\mathbf{1}, 1)$$
   is only possible if $\tilde{\psi}_i > 0$ for $i = 1, \ldots, m$ because $X_+ \cap H(\mathbf{1}, 1)$ contains all the state securities.

   (d) Finally,
   $$\tilde{\psi}^* y = 0 \text{ for all } y \in M_0,$$
   implies the statement
   $$\text{for any } x \in \mathbb{R}^n,\ S^* x = 0 \Rightarrow \tilde{\psi}^* A x = 0,$$
   which is only possible if there is a constant c such that
   $$c S^* = \tilde{\psi}^* A. \tag{2}$$

(e) By virtue of (1) one must also have

$$\text{for any } x \in \mathbb{R}^n,\ S^*x < 0 \Rightarrow \tilde{\psi}^* Ax \leq 0,$$

which can be written as

$$\text{for any } x \in \mathbb{R}^n,\ S^*x < 0 \Rightarrow cS^*x \leq 0,$$

which can only hold either if S is identically a zero vector (and we can choose $c = 1$) or for c in (2) we have $c > 0$. In either case $\psi := \tilde{\psi}/c$ is a strictly positive state price vector solving

$$A^*\psi = S.$$

2. The "only if" part. Suppose we have a strictly positive state price vector such that $S^* = \psi^* A$. Then

$$(Ax \geq \mathbf{0}) \wedge (Ax \neq \mathbf{0}) \Rightarrow \psi^* Ax > 0 \Rightarrow S^*x > 0$$

which means no arbitrage is possible.

3. The arbitrage theorem is known in mathematics as the Farkas' lemma, see Beavis and Dobbs (1990) for a very short proof.

4. The separation theorem is depicted below for an incomplete market with $n = 2$ assets and $m = 3$ states, see Figures 1 and 2 below.

   (a) Refer to Figure 1 below. In financial terminology the claims with zero cost (M_0) correspond to excess returns (i.e. the difference between the return of a risky asset and a benchmark asset; the benchmark is often taken to be the safe asset).

   (b) Refer to Figure 2 below. Any incomplete market without arbitrage can be completed by adding fictitious securities in such a way that the completed market has no arbitrage. Excess returns in the completed market span a whole plane. This plane has a unique (up to scaling) vector orthogonal to it. In other words, the plane can be chosen so as not to intersect the positive orthant, in which case the corresponding normal vector *must* actually pass through the positive orthant (implying $\psi > 0$). The correct length of ψ is found by pricing a strictly positive claim (typically a risk-free bond, yielding $\sum_{i=1}^m \psi_i = 1/R_{\mathrm{f}}$).

References

Beavis, B. and I. Dobbs (1990). *Optimization and Stability Theory for Economic Analysis*. Cambridge University Press.

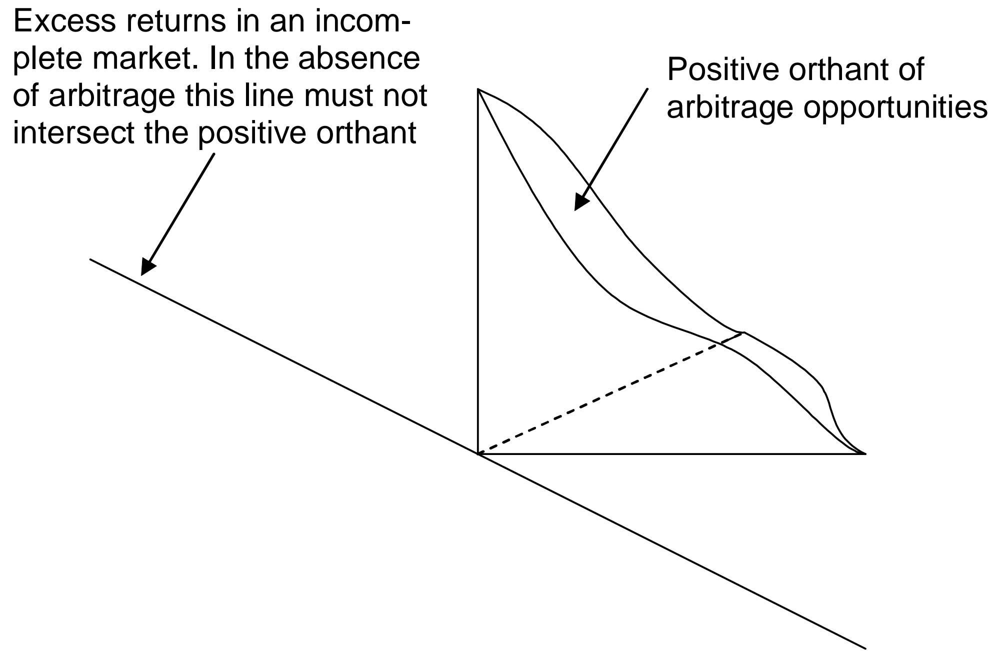

Figure 1: No arbitrage in an incomplete market.

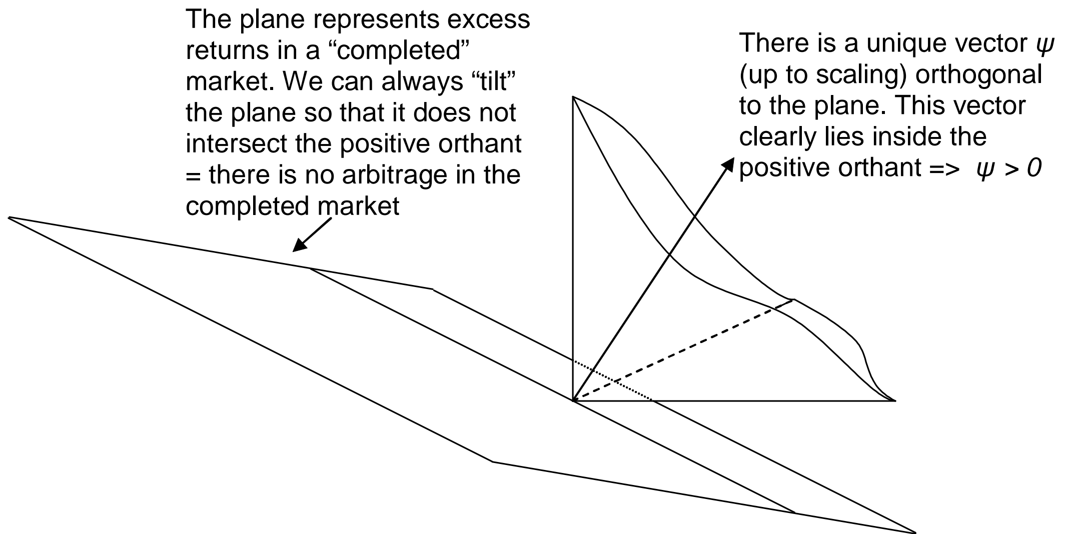

Figure 2: Completed market permitting no arbitrage.