13.8 Set

$$\mu = \mathrm{E}\left[X\right], \qquad \sigma^2 = \mathrm{E}\left[(X-\mu)^2\right].$$

In the first step we will simplify $\mathrm{E}\left[(X-\mu)^4\right]$ and $\mathrm{E}\left[(X-\mu)^3\right]$ so that we can express the non-central moments $\mathrm{E}\left[X^4\right]$ and $\mathrm{E}\left[X^3\right]$ in terms of the central moments $\mathrm{E}\left[(X-\mu)^4\right]$ and $\mathrm{E}\left[(X-\mu)^3\right]$.

$$\begin{aligned}\mathrm{E}\left[(X-\mu)^3\right] &= \mathrm{E}\left[X^3 - 3\mu X^2 + 3\mu^2 X - \mu^3\right] \\ &= \mathrm{E}\left[X^3\right] - 3\mu\left(\mu^2+\sigma^2\right) + 2\mu^3 \\ &= \mathrm{E}\left[X^3\right] - 3\mu\sigma^2 - \mu^3,\end{aligned}$$

and therefore,

$$\mathrm{E}\left[X^3\right] = \mathrm{E}\left[(X-\mu)^3\right] + 3\mu\sigma^2 + \mu^3.$$

For the fourth central moment we have

$$\begin{aligned}\mathrm{E}\left[(X-\mu)^4\right] &= \mathrm{E}\left[X^4 - 4\mu X^3 + 6\mu^2X^2 - 4\mu^3X + \mu^4\right] \\ &= \mathrm{E}\left[X^4\right] - 4\mu\mathrm{E}\left[X^3\right] + 6\mu^2\left(\mu^2+\sigma^2\right) - 3\mu^4 \\ &= \mathrm{E}\left[X^4\right] - 4\mu\mathrm{E}\left[X^3\right] + 3\mu^4 + 6\mu^2\sigma^2,\end{aligned}$$

and consequently,

$$\begin{aligned}\mathrm{E}\left[X^4\right] &= \mathrm{E}\left[(X-\mu)^4\right] + 4\mu\mathrm{E}\left[X^3\right] - 3\mu^4 - 6\mu^2\sigma^2 \\ &= \mathrm{E}\left[(X-\mu)^4\right] + 4\mu\left(\mathrm{E}\left[(X-\mu)^3\right] + 3\mu\sigma^2 + \mu^3\right) - 3\mu^4 - 6\mu^2\sigma^2 \\ &= \mathrm{E}\left[(X-\mu)^4\right] + 4\mu\mathrm{E}\left[(X-\mu)^3\right] + 6\mu^2\sigma^2 + \mu^4. \qquad (1)\end{aligned}$$

Now substitute (1) into the expression for $\mathrm{Var}(X^2)$

$$\begin{aligned}\mathrm{Var}(X^2) &= \mathrm{E}\left[X^4\right] - \left(\mathrm{E}\left[X^2\right]\right)^2 = \mathrm{E}\left[X^4\right] - \left(\sigma^2+\mu^2\right)^2 \\ &= \mathrm{E}\left[(X-\mu)^4\right] + 4\mu\mathrm{E}\left[(X-\mu)^3\right] + 4\mu^2\sigma^2 - \sigma^4 \\ &= \sigma^4\left(\frac{\mathrm{E}\left[(X-\mu)^4\right]}{\sigma^4} - 1 + 4\frac{\mu}{\sigma}\frac{\mathrm{E}\left[(X-\mu)^3\right]}{\sigma^3} + 4\left(\frac{\mu}{\sigma}\right)^2\right) \\ &= \sigma^4\left(\mathrm{Kurt}(X) - 1 + 4\frac{\mu}{\sigma}\mathrm{Skew}(X) + 4\left(\frac{\mu}{\sigma}\right)^2\right).\end{aligned}$$